ENG1001
Engineering Mathematics

Chapter 1. First-Order ODEs

Contents:

Questions?: First-Order

ODEs: Ordinary Differential Equations

1.1 Basic Concepts

■ Ordinary Differential Equation (ODE):

Contains one or several derivatives of an unknown function

one independent variable

Ex) unknown function $y(x)$

(1) $y' \cos x,$

(2) $y'' + 9y = 0$

(3) $y'y''' - \frac{3}{2}y'^2 = 0$

Note: $y' = \frac{dy}{dx}$: derivative

▶ ODE vs PDE

tips: ODE-상미분방정식, PDE-편미분방정식

■ Partial Differential Equation (PDE):

Involves an unknown function of two or more independent variables and its partial derivatives

More complex than ODE (will be covered in Chapter 12)

Ex) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

It is called Laplace Equation (Soil Mechanics)

To get Abeek Credit, Engineering Mathematics II is mandatory.

Order of Differential Equation:

The highest derivative that appears in the equations

(1) $y' = \cos x,$ $\Rightarrow$ 1st order

(2) $y'' + 9y = 0,$ $\Rightarrow$ 2nd order

(3) $y'y''' - \frac{3}{2}y'^2 = 0$ $\Rightarrow$ 3rd order

First-order (Ordinary) Differential Equation (p.4):

Consists of unknown function (y), its derivative (y'), and variable (x)

$\Rightarrow$ $F(x, y, y') = 0$, or $y' = f(x, y)$

- ▶ Implicit solution: $F(x, y, y') = 0$
- ▶ Explicit solution: $y' = f(x, y)$

tips: order n – derivative의 최대 차수

Concept of Solution (p.4)

A function y=h(x) is called a solution of a given first-order differential equation F(x, y, y') = 0, on some open interval a<x<b, if h(x) is defined and differential throughout the interval

▶ Explicit solution : $y' = f(x, y)$

Ex) $xy' = 2y$ $\Rightarrow$ $y = x^2$

▶ Implicit solution : $F(x, y, y') = 0$

Ex) $yy' = -x$ $\Rightarrow$ $x^2 + y^2 = 1$ $(x^2 + y^2 - 1 = 0 \ (y > 0, -1 < x < 1))$

Concept of Solutions (Continued)

- General Solution: contains an arbitrary constant (c)

Ex) $y' = \cos x \quad \Rightarrow \quad y = \sin x + c$

- Particular Solution: contains a specific constant (c = 2)

Ex) $y' = \cos x \quad \Rightarrow \quad y = \sin x + 2$

tips: general solution – 일반해
particular solution – 특해

Modeling (p.6)

Applications : Many physical laws and principles appear mathematically in the form of differential equations

Initial value problems : differential equations with an initial condition
obtain particular solution from general solution

$$y' = f(x, y), \quad y(x_0) = y_0$$

Modeling : physical systems ⇒ mathematical formulation and analysis
(자연현상의 문제를 Equation으로)

Step1. Setting up a mathematical model
Step2. General Solution
Step3. Particular Solution
Step4. Check the results

Ex. 5 (p.7): Radioactivity, Exponential Decay

Given:
1. Decomposition rate of a radioactive substance is proportional to the amount present.
2. Initially, 0.5 gram of radioactive substance is present.

Solution:
Unknown Function: *y(t)*
(y = the amount of a radioactive substance, t = time)

Step 1: Setting up a mathematical model of the physical process (modeling)

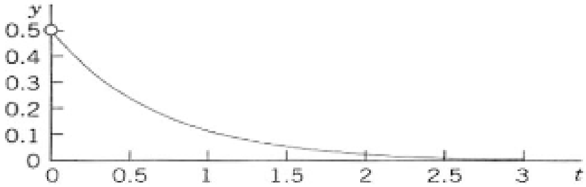
1. :

2. :

Step 2: Solving Differential Equation (General Solution)

Step 3: Determination of a particular solution from an initial condition

Step 4: Checking



Radioactivity (Exponential decay, $y = 0.5\,e^{kt}$, with $k = -1.5$ as an example)

1.3 Separable ODEs. Modeling

** Separable vs non-separable ⇒ reduce to separable form

■ Separable Differential Equations:

Separate the variables x (on the right) and y (on the left)

tips: 분리해서 푼다 (x는 x대로, y는 y대로)

■ Solution: Integrate on both sides with respect to x

⇒ if *f* and *g* are continuous functions, the integrals will exist.

■ Solving Procedure

Step 1. Separate the variables

Step 2. Integrate both sides

Step 3. Rearrange in explicit (or implicit) forms

Ex. 1 (p. 12) $y' = 1 + y^2$

Ex. 3 (p. 13): Initial Value Problem $y' = -2xy, \quad y(0) = 1.8$

$$\frac{y'}{y} = -2x \quad \Rightarrow \quad \frac{dy/dx}{y} = -2x \quad \Rightarrow \quad \frac{dy}{y} = -2x\,dx \qquad \text{(Step 1)}$$

$$\Rightarrow \quad \int \frac{1}{y}dy = \int -2x\,dx + c \quad \Rightarrow \quad \ln|y| = -x^2 + c^* \qquad \text{(Step 2)}$$

$$\Rightarrow \quad y = e^{-x^2 + c^*} = ce^{-x^2} \qquad \text{(Step 3)}$$

$$y(0) = ce^0 = c = 1.8 \qquad \text{(Initial Value)}$$

$$\Rightarrow y = 1.8e^{-x^2}$$

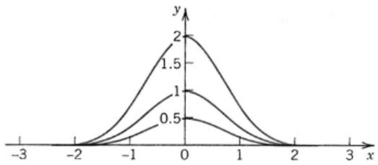


Fig. 8. Solutions of $y' = -2xy$ ("bell-shaped curves") in the upper half-plane

Normal distribution

■ Modeling (p.13):

Natural or Engineered Systems ⇒
Mathematical Formulation and Analysis

Ex 5. (p.14): Mixing Problem (Environmental, Geoenvironmental, Hydrology)
The tank contains 1000 gal of water in which initially 100 lb of salt are dissolved. Brine runs in at a rate of 10gal/min, and each gallon contains 5 lb of dissolved salt.
The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal/min.
Find the amount of salt in the tank at any time t.

Step 1: Setting up a Model
y: the amount of salt in the tank at time t
Mass (Salt) balance: Rate of salt change (dy/dt=y') = Inflow salt – Outflow salt

Inflow salt =
Outflow salt =

$\Rightarrow$

Step 2: Solution of Model (Separable Variables)

$$y' = -0.01(y-5000) \Rightarrow \frac{y'}{y-5000} = -0.01 \Rightarrow \frac{dy}{y-5000} = -0.01dt$$

$$\int \frac{1}{y-5000}dy = \int -0.01dt + c^* \Rightarrow \ln|y-5000| = -0.01t + c^*$$

$$\Rightarrow y - 5000 = e^{-0.01t}e^{c^*} = ce^{-0.01t}$$

$$\Rightarrow y = 5000 + ce^{-0.01t}$$

Step 3: Determination of a particular solution from an initial condition

$$y(0) = 5000 + ce^{0} = 5000 + c = 100 \Rightarrow c = -4900$$

$$y = 5000 - 4900e^{-0.01t}$$

$$y = 5000 - 4900e^{-0.01t}$$

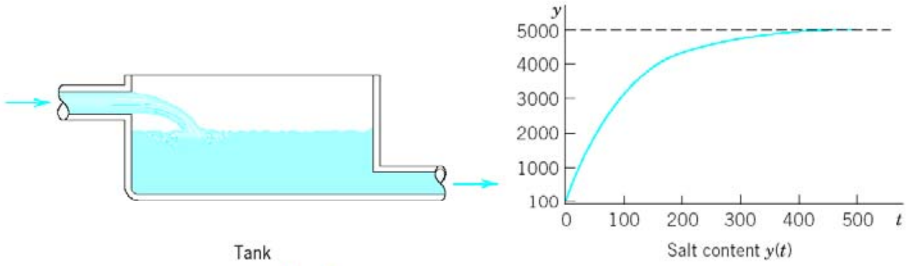


Fig. 9. Mixing problem in Example 3

■ Extended Method: Reduction to Separable Form (p.17)

Certain differential equations are not separable, but can be made separable by the introduction of a new unknown function.

Differential equation *y'=f(y/x)* such as $\sin(\frac{y}{x})$

Set $u = y/x$

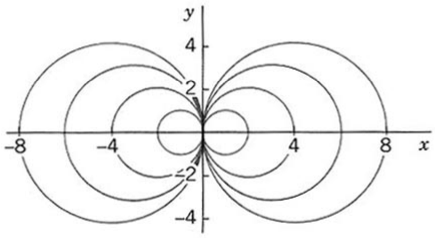
$$\Rightarrow\ y = ux \ \Rightarrow\ y' = (ux)' \qquad \text{(introduction of } \boldsymbol{u}\text{)}$$

$$y' = f(y/x) \Rightarrow u'x + u = f(u) \Rightarrow u'x = f(u) - u \Rightarrow \frac{du}{f(u)-u} = \frac{dx}{x}$$

$$\Rightarrow \int \frac{1}{f(u)-u} du = \int \frac{1}{x} dx + c$$

Ex 8. (p.18): $2xyy' = y^2 - x^2$

$$2xyy' = y^2 - x^2 \;\Rightarrow\; y' = \frac{1}{2}\left(\frac{y}{x} - \frac{x}{y}\right)$$



General solution (family of circles) in Example 6

A family of circles passing through the origin with centers on the x-axis

1.4 Exact ODEs. Integrating Factors

Exact vs Nonexact

Integrating Factor

- Exact Differential Equation

$$u(x,y)=c \Rightarrow du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

$$\frac{\partial u}{\partial x} = M(x,y), \quad \frac{\partial u}{\partial y} = N(x,y)$$

$$M(x,y)dx + N(x,y)dy = 0$$

- Conditions (*necessary and sufficient*) for Exact Differential Equation

$$M(x,y) = \frac{\partial u}{\partial x} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x \partial y}$$

$$N(x,y) = \frac{\partial u}{\partial y} \Rightarrow \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}$$

$$\Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{(Test for Exactness)}}$$

Solution of Exact Differential Equation

$$M(x,y)=\frac{\partial u}{\partial x}$$

$$\Rightarrow u(x,y)=\int M(x,y)dx+k(y)=c$$

$$\frac{\partial u}{\partial y}=\_\_\_\_\_\_+\frac{\partial k(y)}{\partial y}=N$$

$$to\ \ find\ \ k(y)$$

$$N(x,y)=\frac{\partial u}{\partial y}$$

$$\Rightarrow u(x,y)=\int N(x,y)dy+l(x)=c$$

$$\frac{\partial u}{\partial x}=\_\_\_\_\_\_+\frac{\partial l(x)}{\partial x}=M$$

$$to\ \ find\ \ l(x)$$

Ex 1. (p.22): An Exact ODE

Solve $\cos(x+y)dx+(3y^2+2y+\cos(x+y))dy=0$

Reduction to Exact Form. Integrating Factors (p.23)

If the differential equation is not exact,
make it to be exact by multiplying *F(x,y)* integrating factor

$$M(x,y)dx + N(x,y)dy = 0 \qquad \text{Exact}$$

$$P(x,y)dx + Q(x,y)dy = 0 \qquad \text{Non-exact}$$

$$\Rightarrow \; F(x,y)P(x,y)dx + F(x,y)Q(x,y)dy = 0 \qquad \text{Exact}$$

Ex 3. (p. 23): $-ydx + xdy = 0$

$$M = -y, \qquad N = x$$

$$\partial M / \partial y = -1, \quad \partial N / \partial x = 1 \quad \Rightarrow \quad \text{Non-exact}$$

By multiplying by $1/x^2$ (integrating factor)

$$\Rightarrow -\frac{y}{x^2}dx + \frac{1}{x}dy = 0$$

$$\Rightarrow M = -\frac{y}{x^2}, \quad N = \frac{1}{x}$$

$$\Rightarrow \frac{\partial M}{\partial y} = -\frac{1}{x^2}, \quad \frac{\partial N}{\partial x} = -\frac{1}{x^2} \Rightarrow \text{Exact!!}$$

How to Find Integrating Factors (p.24)

$FPdx + FQdy = 0$ (Exact Differential Equation)

$$\Rightarrow \frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ)$$

$$\Rightarrow \frac{\partial F}{\partial y}P + F\frac{\partial P}{\partial y} = \frac{\partial F}{\partial x}Q + F\frac{\partial Q}{\partial x}$$

$$\Rightarrow F_y P + F P_y = F_x Q + F Q_x \quad (F_x = dF/dx,\ F_y = dF/dy, \ldots)$$

Golden Rule:
Look for an integrating factor depending only on one variable (x or y)

▶ Case 1: Integrating factor *F(x)* (Theorem 1)

Since $F_y = dF/dy = 0$

$$\Rightarrow F_yP + FP_y = F_xQ + FQ_x \Rightarrow FP_y = F'Q + FQ_x \quad (F_x = F')$$

Dividing by FQ

$$\Rightarrow \frac{P_y}{Q} = \frac{F'}{F} + \frac{Q_x}{Q} \Rightarrow \frac{1}{F}\frac{dF}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$\boxed{\text{Set} \quad R(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}$$

? $\frac{1}{F}\frac{dF}{dx}$ $R(x)$? $\ln|F|$ 室$R(x)dx + c$? $\boxed{F(x) = \exp\left(\quad R(x)dx\right)}$

Separable ODE (sec 1.3)

Summary of Integrating Factor

▶ Case 1: Integrating factor *F(x)* (Theorem 1)

Set $R(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right).$

$$\Rightarrow \frac{1}{F}\frac{dF}{dx} = R(x) \Rightarrow \ln|F| = \int R(x)dx + c \Rightarrow \therefore \boxed{F(x) = \exp\left(\int R(x)dx\right)}$$

▶ Case 2: Integrating factor *F(y)* (Theorem 2)

Similar to $F(x)$

$$R^*(y) = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \qquad \boxed{F^*(y) = \exp\left(\int R^*(y)dy\right)}$$

Ex. 5 (p.24): $(e^{x+y} + ye^{y})dx + (xe^{y} - 1)dy = 0, \quad y(0) = -1$

use Theorem1 or Theorem2
Question: Find integrating factor and Solve the initial value problem

1) Check the Exact ODE

(1) $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(e^{x+y} + ye^{y}) =$

(2) $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(xe^{y} - 1) \quad =$

2) Theorem

(1) Theorem 1: $R = \frac{1}{Q}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}) = \frac{1}{xe^{y} - 1}(e^{x+y} + e^{y} + ye^{y} - e^{y})$

(2) Theorem 2: $R^{*} = \frac{1}{P}(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) = \frac{1}{e^{x+y} + ye^{y}}(e^{y} - e^{x+y} - e^{y} - ye^{y}) = -1$

3) Integrating factor

4) Exact ODE

5) Integration

6) Differentiate to get k(y)

7) Solution

8) Initial Value Problem

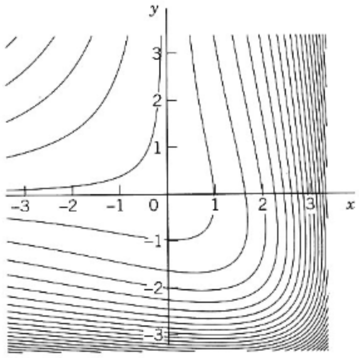


Fig. 15. Particular solutions in Example 5

1.5 Linear ODEs. Bernoulli Equation

Homogeneous vs Nonhomogeneous

⇒ Exact ODE (Sec. 1.4)

Linear vs Nonlinar

⇒ Linearization

Linear (First Order) Differential Equation

$$y' + p(x)y = r(x)$$

unknown function y and its derivative y' : Linear

p(x): function of x

$y' + p(x)y = 0$	$\Rightarrow$ Homogenous Linear ODE		$r(x) = 0$
$y' + p(x)y = r(x) \neq 0$	$\Rightarrow$ Nonhomogenous Linear ODE		$r(x) \neq 0$

▶ Solution of Homogeneous Linear (First Order) Differential Equation

$$y' + p(x)y = 0 \Rightarrow \frac{dy}{dx} = -p(x)y \Rightarrow \frac{dy}{y} = -p(x)dx$$

$$\Rightarrow \ln|y| = -\int p(x)dx + c^*$$

$$\Rightarrow \boxed{y = ce^{-\int p(x)dx}}$$

if c=0, y=0 trivial solution

▶ Solution of Non-homogeneous Linear (First Order) Differential Equation

if exact: good

if non-exact ⇒ integrating factor ⇒ exact ODE

$$y' + p(x)y = r \Rightarrow (py - r)dx + dy = 0 \Rightarrow Pdx + Qdy = 0$$

$$P = (py - r), \quad Q = 1 \Rightarrow \text{non-exact}$$

Need to find integrating factor $F(x)$ (Theorem 1)

$$R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = p \Rightarrow F = \exp\left(\int R dx\right) \Rightarrow \boxed{F = e^{\int pdx}}$$ (slide 30)

Multiplying by integrating factor

$$e^{\int pdx}(y' + py) = e^{\int pdx} r \Rightarrow e^{\int pdx} y' + y\left(e^{\int pdx}\right)' = \left(e^{\int pdx} y\right)' = e^{\int pdx} r$$

$$\Rightarrow e^{\int pdx} y = \int e^{\int pdx} r dx + c$$

$$\Rightarrow \boxed{y = e^{-h}\left(\int e^{h} r dx + c\right), \quad h = \int pdx}$$

■ Input and Output (p.29)

$$y' + p(x)y = r(x)$$

$r(x)$: input,

$y(x)$: output

$$\boxed{y(x) = e^{-h}\int e^{h} r\,dx + ce^{-h}, \qquad h = \int p\,dx}$$

Total Output = Response to Input (*r*) + Response to Initial Data (*c*)

Ex 1.: $y' - y = e^{2x}$

■ Linear vs Nonlinear ➔ Reduction to Linear Form (p.31)

Bernoulli Equation : $y' + p(x)y = g(x)y^{a}$ (a: any real number)

$a = 0 \;\; or \;\; a = 1$: linear

$a \neq 0 \;\; and \;\; a \neq 1$: nonlinear

Transform to Linear Differential Equation

$$y' + p(x)y = g(x)y^{a} \quad \Rightarrow \quad y' = g(x)y^{a} - p(x)y$$

Ex 4. (p.32): Logistic Equation

$$y' - Ay = -By^2$$

$$y' + p(x)y = g(x)y^a \quad \Rightarrow \quad a = 2 \quad u = y^{-1}$$

$$\Rightarrow \ p = A, \quad r = B, \quad h = \int p\,dx = Ax$$ (Slide 38)

$$\Rightarrow \ u = e^{-h}\left(\int e^h r\,dx + c\right) = e^{-Ax}\left(\frac{B}{A}e^{Ax} + c\right) = ce^{-Ax} + B/A$$

$$\Rightarrow \ \therefore \ y = \frac{1}{u} = \frac{1}{(B/A) + ce^{-Ax}}$$

▶ Summary of the Solution of (First Order) Linear Differential Equation

$$y' + p(x)y = r(x) \ \Rightarrow \ y = e^{-h}\left(\int e^h r\,dx + c\right), \quad h = \int p\,dx$$

When *p = 0, r = 0*	:	$y' = 0$	$\Rightarrow$	$y = c$
When *r=0*	:	$y' + py = 0$	$\Rightarrow$	$y = ce^{-\int p(x)dx}$
When *p=0*	:	$y' = r$	$\Rightarrow$	$y = \int r\,dx + c$

1.6 Orthogonal Trajectories

■ **Orthogonal Trajectories**

⇒ **Family of Curves at right angle**

(1) G(x,y,c)=0

ex) $x^2 + 2y^2 - 2c^2 = 0 \qquad \therefore \frac{1}{2}x^2 + y^2 = c^2$

tips: orthogonal – perpendicular – right angle

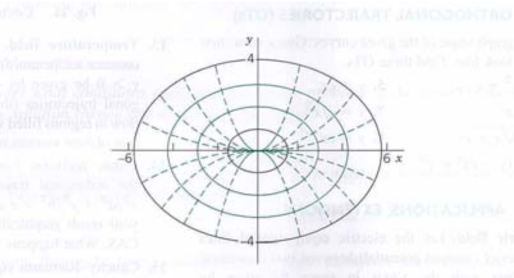
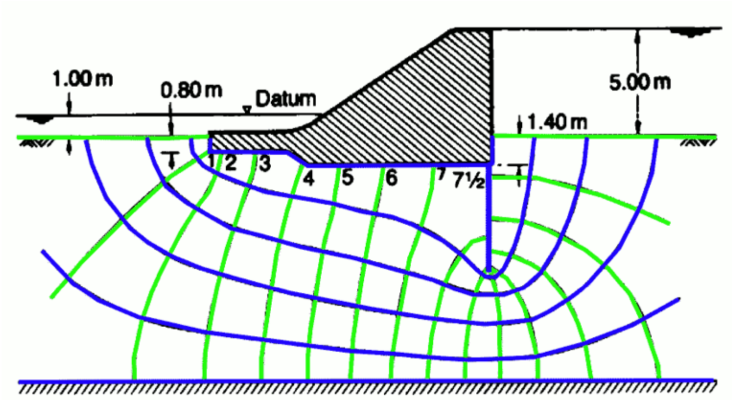


Fig. 24. Electrostatic field between two ellipses (elliptic cylinders in space): Elliptic equipotential curves (equipotential surfaces) and orthogonal trajectories (parabolas)

Procedures

Seepage (Soil Mechanics).



Summary – Chapter 1

$$y' + p(x)y = r(x)$$

- General Solution: contains an arbitrary constant (c) $\quad y = \sin x + c$
- Particular Solution: contains a specific constant (c = 2) $\quad y = \sin x + 2$

- Separable: Separate the variables x and y

$$g(y)y' = f(x) \quad \Rightarrow \quad g(y)dy = f(x)dx \quad (\because y' = \frac{dy}{dx})$$

  - Non-separable ⇒ reduce to separable form

- Exact ODE $\quad M(x,y)dx + N(x,y)dy = 0 \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
  - Non-exact ⇒ reduce to exact form – integrating factor

- Homogeneous and Linear: Separable or Exact ODE

- Nonlinear
  - Nonlinear ⇒ reduce to linear form

Homework Assignment – Chapter 1

Assignment # 1-1

Due: Next week

Section 1.1: 6, 10, 12

Section 1.3: 2, 8, 12, 17

Assignment # 1-2

Due: Next week

Section 1.4: 4, 9, 13

Section 1.5: 4, 10, 23, 24

Section 1.6: 9

Announcement: Quiz

March 20: Chapter 1

Closed book

Homework Assignment – Chapter 1

1.4 Example : Quiz or Mid-term exam

Example. $(x^3+3xy^2)dx+(3x^2y+y^3)dy=0$

Step 1: Test for Exactness

Step 2: Solving Exact Differential Equation

In order to find $k(y)$

Step 3: Checking

1.4 Quiz or Mid-term Exam

Example. $2\sin(y^2)dx + xy\cos(y^2)dy = 0, \quad y(2) = \sqrt{\pi/2}$

Step 1: Test for Exactness

$$P = 2\sin(y^2), \qquad Q = xy\cos(y^2)$$

$$\Rightarrow P_y = 4y\cos(y^2) \qquad Q_x = y\cos(y^2)$$ → non-exact

Step 2: Find integrating factor $F(x)$

$$R = \frac{1}{Q}(P_y - Q_x) = \frac{1}{xy\cos(y^2)}[4y\cos(y^2) - y\cos(y^2)] = \frac{3y}{xy} = \frac{3}{x}$$

$$\Rightarrow \int R(x) = \int \frac{3}{x}dx = 3\ln|x|$$

$$\Rightarrow \therefore F(x) = \exp(\int Rdx) = \exp(3\ln|x|) = \exp(\ln|x^3|) = x^3$$

Multiplying by $F(x) = x^3$

$$FPdx + FQdy = 2x^3\sin(y^2)dx + x^4 y\cos(y^2)dy = 0$$

$$\frac{\partial}{\partial y}[2x^3\sin(y^2)] = 4x^3 y\cos(y^2) = \frac{\partial}{\partial x}[x^4 y\cos(y^2)]$$: Exact !!

Step 3: Solve Exact Differential Equation (General solution)

$$M = u_x = 2x^3\sin(y^2), \qquad N = u_y = x^4 y\cos(y^2)$$

$$\Rightarrow\ u = \frac{x^4}{2}\sin(y^2) + k(y), \qquad u_y = x^4 y\cos(y^2) + \frac{dk}{dy}, \qquad \frac{dk}{dy} = 0, \qquad k = c^*$$

$$\Rightarrow\ \therefore u = \frac{x^4}{2}\sin(y^2) = c$$

Step 4: Find particular solution

Initial condition $y(2) = \sqrt{\pi/2}$

$$\Rightarrow\ 8\sin(\pi/2) = 8 = c$$

$$\Rightarrow\ x^4\sin(y^2) = 16$$

1.6 Orthogonal Trajectories

■ **Orthogonal Trajectories**

⇒ **Family of Curves at right angle**

(1) G(x,y,c)=0

ex) $x^2 + y^2 - c^2 = 0 \qquad \therefore x^2 + y^2 = c^2$

tips: orthogonal – perpendicular – right angle

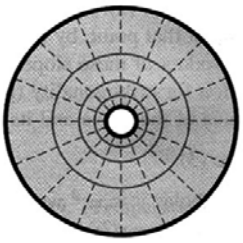


Fig. 22. Equipotential lines and curves of electric force (dashed) between two concentric (black) circles (cylinders in space)

Procedures

y

1) y' Slope

2) $\tilde{y}' = -\dfrac{1}{y'}$ ($y'\tilde{y}' = -1$ $\therefore$ orthogonal)

3) $\tilde{y}$

4) plot

Example (p.35)

$y = cx^2$ $\therefore G(x, y, c) = y - cx^2 = 0$

step1.

step2.

step3.

step4.

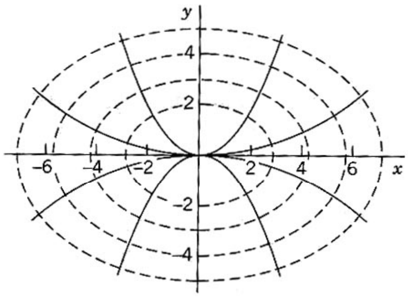


Fig. 23. Parabolas and orthogonal trajectories (ellipses)